

Queen Mary, University of London

B.Sc. Examination by Course Units 2004

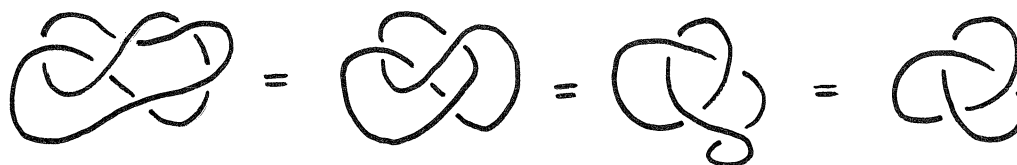
MAS 231 GEOMETRY II: KNOTS & SURFACES

Date and time: 4th May 2004 at 10.00 am

Duration: 2 hours

You may attempt as many questions as you wish and all questions carry equal marks. Except for the award of a bare pass, only the best 4 questions answered will be counted. Calculators are NOT permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

1. (a) Describe the three types of Reidemeister moves and mark which move is used and where in each of the following manipulations of knot diagrams:



(b) State what it means for a knot diagram to be *tricolourable* and prove that tricolourability is invariant under moves of type II. State whether or not tricolourability is invariant under all three moves and hence a property of the knot.

(c) Show that the knot depicted in part (a) is tricolourable.

(d) State the definition of the *writhe* W of a knot diagram and show that

$$W(\boxed{\text{diagram}}) = W(\text{diagram}) - 1$$

where the box denotes the remaining part of any knot diagram.

2. (a) Give an inductive definition of the Kauffman bracket B of a knot or link diagram and show that

$$B(\boxed{\text{diagram}}) = -x^3 B(\text{diagram})$$

where the box denotes the remaining part of any diagram.

Please turn over

(c) Define the Jones knot invariant J in terms of B and the writhe W and show that it is invariant under the transformation of knot diagrams occurring in part (a). You may quote any properties of W that are needed.

(d) Show that

$$B(\text{figure}) = -x^4 - x^{-4}, \quad J(\text{figure}) = t^{-1} + t^{-3} - t^{-4}$$

where $t = x^4$. You may assume that W of the knot diagram on the right is -3.

3. (a) Define what it means for a curve to be *regular*. Sketch the curve

$$\gamma(t) = (\sinh(t), \cosh(t), 0), \quad t \in (-\infty, \infty)$$

and say if it is regular. Show that the curve approaches the lines $y = \pm x$ in the $x - y$ -plane as $t \rightarrow \pm\infty$, and include this limiting behaviour in your sketch. You may use the identity $\cosh(t) = \sqrt{1 + \sinh^2(t)}$.

(b) Define the *signed curvature* K_S of a regular planar curve in the $x - y$ -plane and show that

$$K_S = \frac{1}{(1 + 2\sinh^2(t))^{3/2}}$$

for the curve in part (a).

(c) Hence, or otherwise, show that $\int_{-\infty}^{\infty} \frac{dt}{1 + 2\sinh^2(t)} = \frac{\pi}{2}$. You may assume the result that $K_S ||\dot{\gamma}|| = \dot{\theta}$, the anticlockwise rate of change of the angle of the direction of the curve.

4. (a) Define what it means for a curve to be *unit speed*. Sketch the curve

$$\gamma(t) = \frac{1}{\sqrt{2}}(\cos(t), \sin(t), t)$$

and show that it is unit speed.

(b) Show that for a unit speed curve, the vectors $\tau = \dot{\gamma}$, $\nu = \frac{\ddot{\gamma}}{K}$ and $\beta = \tau \times \nu$ form an orthonormal triple. You may assume that $K = ||\ddot{\gamma}||$ in the unit speed case.

(c) Compute and show the three vectors in part (b) in your sketch at a typical point of the curve in part (a).

(d) Find the torsion T for the curve in part (a) and say if the curve is planar. You may assume that $\dot{\beta} = -T\nu$.

Please turn over

5. (a) Sketch and describe the surface S given by the single surface patch

$$\sigma(u, v) = (u, v, \sqrt{u^2 + v^2}), \quad U = \{(u, v) \in \mathbb{R}^2 ; (u, v) \neq (0, 0)\}.$$

(b) Find the standard unit normal N_S for the surface in part (a) and show it on your sketch at a typical point. Is the surface orientable?

(c) Find the first and second fundamental forms of the surface in part (a) and show that the Gauss curvature vanishes.

6. (a) State the definition of a geodesic on a surface with unit normal N_S . Show that γ is a geodesic if and only if the geodesic curvature $K_g = \ddot{\gamma} \cdot (N_S \times \dot{\gamma})$ vanishes at all points of the curve. You may assume that γ has unit speed.

(b) Consider the torus with surface patch and standard unit normal

$$\begin{aligned} \sigma(\theta, \phi) &= ((a + b \cos(\theta)) \cos(\phi), (a + b \cos(\theta)) \sin(\phi), b \sin(\theta)), \\ N_S(\theta, \phi) &= -(\cos(\theta) \cos(\phi), \cos(\theta) \sin(\phi), \sin(\theta)). \end{aligned}$$

On a sketch of the torus, show the directions of N_S along: (i) the outer circle $\theta = 0$; (ii) the inner circle $\theta = \pi$; (iii) the circle at $\phi = 0$; and (iv) the circle at $\phi = \pi$. Give an argument to show that all four circles are geodesics.

(c) State the Gauss-Bonnet theorem for a curvilinear polygon on a surface S . Explain why this implies that $\int \int_R K_G dA_\sigma = 0$ when R is the region given by $0 < \theta < \pi, 0 < \phi < \pi$. You may assume that the curves (i)-(iv) in part (b) cross each other at right angles.

End of Exam Paper